Journal of Statistical Physics, Vol. 75, Nos. 1/2, 1994

Anderson Localization and the Space-Time Characteristic of Continuum States

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Received November 15, 1993

A proof of Anderson localization is obtained by ruling out any continuous spectrum on the basis of the space-time characteristic of its states.

KEY WORDS: Anderson localization.

1. INTRODUCTION AND RESULTS

By now many proofs[1,4,5,11] of localization for the d-dimensional Anderson model have been given. Common to all of them, as to this one, is the derivation of some form of exponential decay[8] of the Green's function. In a second step, localization, i.e., absence of continuous spectrum, is then obtained either[1,4,11] using results about the behavior of the spectral measure under rank-one perturbations, or[4,5,7] using that the set of generalized eigenvalues has full spectral measure. The purpose of this paper is to show that this step can be done using a characterization of the continuous spectrum due to Ruelle[10] and Amrein and Georgescu.[2,6] Such a possibility has been conjectured in ref. 8. For the one-dimensional model it has been used in refs. 3 and 9.

We consider just the simplest case. This is the discrete Schrödinger operator

$$h_\omega = -\Delta + v_\omega$$

acting on $l^2(\mathbb{Z}^d)$, where Δ is the discrete Laplacian

$$(\Delta\psi)(x) = \sum_{|e|=1} \psi(x+e)$$

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0022-4715/94/0400-0337$07.00/0

and v_ω is a random potential

$$(v_\omega \psi)(x) = v_x \psi(x)$$

Here $|x| = \sum_{i=1}^{d} |x_i|$ and $\omega = \{v_x\}_{x \in \mathbb{Z}^d}$ is a collection of independent identically distributed random variables. We shall assume that the single-site probability distribution has a density $\rho \in L^1(\mathbb{R})$, $\|\rho\|_1 = 1$, with respect to Lebesgue measure. In other words, the probability space $\Omega = \times_{x \in \mathbb{Z}^d} \mathbb{R}$ is equipped with the probability measure $dP(\omega) = \prod_{x \in \mathbb{Z}^d} \rho(v_x)\, dv_x$.

The strength of the disorder is measured by $\|\rho\|_\infty^{-1}$. Localization occurs if the disorder is large enough.

Theorem 1. Let $\rho \in L^\infty(\mathbb{R})$ and of compact support. If $\|\rho\|_\infty$ is small enough, then h_ω has only pure point spectrum with probability 1.

Ruelle's criterion asserts that states associated with the continuous spectrum leave any compact set in the time mean. More precisely, let E_c be the projection onto the continuous spectral subspace of an operator h on $l^2(\mathbb{Z}^d)$ and let $P_{|x| \geq R}$ be the projection onto wave functions which vanish in $\{x \in \mathbb{Z}^d \mid |x| < R\}$. Then

$$\|E_c \psi\|^2 = \lim_{R \to \infty} \lim_{t \to \infty} \frac{1}{t} \int_0^t ds\, \|P_{|x| \geq R} e^{-ihs} \psi\|^2 \tag{1}$$

$$= \lim_{R \to \infty} \lim_{\varepsilon \downarrow 0} 2\varepsilon \int_0^\infty ds\, e^{-2\varepsilon s} \|P_{|x| \geq R} e^{-ihs} \psi\|^2$$

$$= \lim_{R \to \infty} \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int dE\, \|P_{|x| \geq R} (h - E - i\varepsilon)^{-1} \psi\|^2 \tag{2}$$

The Green's function consists of matrix elements of the resolvent

$$G(x, y; z) = (\delta_x, (h - z)^{-1} \delta_y)$$

where the states δ_n are given by $\delta_n(m) = \delta_{nm}$ $(n, m \in \mathbb{Z}^d)$.

Lemma 2. Let $\|\rho\|_\infty$ be small enough and $0 < s < 1$. Then there are C, $m > 0$ such that

$$\langle |G_\omega(x, y; z)|^s \rangle \leq C e^{-m|x-y|} \tag{3}$$

for all $z \in \mathbb{C} \backslash \mathbb{R}$, $x, y \in \mathbb{Z}^d$.

Here $\langle \cdot \rangle$ denotes the expectation with respect to the probability measure. In ref. 1 a similar estimate was obtained. There the Green's function is regularized by going to finite volumes; here, by going to complex energies.

We then extend the result to higher moments of the Green's function. Specifically:

Lemma 3. Let ρ be as in Lemma 2 and, in addition, of compact support. Then there are C, $m > 0$ such that

$$|\text{Im}\, z| \langle |G_\omega(x, y; z)|^2 \rangle \leqslant C e^{-m|x-y|} \tag{4}$$

for all $z \in \mathbb{C} \backslash \mathbb{R}$, x, $y \in \mathbb{Z}^d$.

Note that the moment of the Green's function in (3) stays bounded as z approaches the real axis, whereas in (4) it may diverge like $|\text{Im}\, z|^{-1}$. Setting $z = E + i\varepsilon$, we shall see that (4) controls the expectation of (2).

The conductivity tensor as defined by the Kubo–Greenwood formula[8] is

$$\sigma_{ij}(E) = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon^2}{\pi} \sum_{x \in \mathbb{Z}^d} x_i x_j \langle |G_\omega(0, x; E + i\varepsilon)|^2 \rangle$$

From (4) we immediately get:

Corollary 4:

$$\sigma_{ij}(E) = 0$$

2. PROOFS

We follow ref. 1 quite closely and begin with:

Lemma 5. Let $0 < s < 1$. Then there is $C > 0$ such that

$$\langle |G_\omega(x, y; z)|^s \rangle \leqslant C \, \|\rho\|_\infty^s \tag{5}$$

for all $z \in \mathbb{C} \backslash \mathbb{R}$, x, $y \in \mathbb{Z}^d$.

Proof. We assume $x \neq y$, the case $x = y$ being similar but easier. The dependence of $G_\omega(x, y; z)$ on v_x, v_y (at fixed values of the potential elsewhere) is particularly simple. To exhibit it, one writes

$$h_\omega = h_{\hat{\omega}} + v_x P_x + v_y P_y$$

where $\hat{\omega}$ is obtained from ω by setting $v_x = v_y = 0$, and $P_n = \delta_n(\delta_n, \cdot)$ are the projections on the states δ_n. Note that h_ω differs from $h_{\hat{\omega}}$ by a rank-2 perturbation acting on the range of $P = P_x + P_y$. From the second resolvent identity $(h_{\hat{\omega}} - z)^{-1} = [1 + (h_{\hat{\omega}} - z)^{-1} (v_x P_x + v_y P_y)](h_\omega - z)^{-1}$ we obtain an identity on Ran P known as Krein's formula:

$$P(h_\omega - z)^{-1} P = (A + v_x P_x + v_y P_y)^{-1} \tag{6}$$

where $A = [P(h_{\hat\omega} - z)^{-1} P]^{-1}$, provided it exists, acts on Ran P and is independent of v_x, v_y. It indeed exists for $z \in \mathbb{C}\backslash\mathbb{R}$ because $(\operatorname{Im} z)^{-1} \operatorname{Im}(h_{\hat\omega} - z)^{-1} = (h_{\hat\omega} - \bar z)^{-1} (h_{\hat\omega} - z)^{-1}$ is positive definite. In particular,

$$-\frac{\operatorname{Im} A}{\operatorname{Im} z} = \frac{1}{\operatorname{Im} z} \frac{A^* - A}{2i} = A^* P \frac{\operatorname{Im}(h_{\hat\omega} - z)^{-1}}{\operatorname{Im} z} PA$$

is positive definite, too. Using matrix notation with respect to the basis $\{\delta_x, \delta_y\}$,

$$A = \begin{pmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{pmatrix}, \qquad \operatorname{Im} A = \begin{pmatrix} \operatorname{Im} a_{xx} & (1/2i)(a_{xy} - \overline{a_{yx}}) \\ (1/2i)(a_{yx} - \overline{a_{xy}}) & \operatorname{Im} a_{yy} \end{pmatrix}$$

we thus have from (6)

$$G_\omega(x, y; z) = -\frac{a_{xy}}{(v_x + a_{xx})(v_y + a_{yy}) - a_{xy} a_{yx}} \tag{7}$$

By retaining only the real, resp. the imaginary part of the denominator, we get

$$|G_\omega(x, y; z)| \leqslant \frac{|a_{xy}|}{|u_x u_y - \operatorname{Im} a_{xx} \operatorname{Im} a_{yy} - \operatorname{Re}(a_{xy} a_{yx})|}$$

$$|G_\omega(x, y; z)| \leqslant \frac{|a_{xy}|}{|u_x \operatorname{Im} a_{yy} + u_y \operatorname{Im} a_{xx} - \operatorname{Im}(a_{xy} a_{yx})|}$$

with $u_i = v_i + \operatorname{Re} a_{ii}$ $(i = x, y)$. Moreover,

$$\det \operatorname{Im} A = \operatorname{Im} a_{xx} \operatorname{Im} a_{yy} + \tfrac{1}{2}\operatorname{Re}(a_{xy} a_{yx}) - \tfrac{1}{4}(|a_{xy}|^2 + |a_{yx}|^2) > 0 \tag{8}$$

(i) We shall first treat the case where

$$\max(|\operatorname{Im} a_{xx}|, |\operatorname{Im} a_{yy}|) < \tfrac{1}{2}|a_{xy}| \tag{9}$$

Using (8), we then have

$$\begin{aligned} c^2 &:= \operatorname{Im} a_{xx} \operatorname{Im} a_{yy} + \operatorname{Re}(a_{xy} a_{yx}) \\ &> \tfrac{1}{2}(|a_{xy}|^2 + |a_{yx}|^2) - \operatorname{Im} a_{xx} \operatorname{Im} a_{yy} > \tfrac{1}{4}|a_{xy}|^2 \end{aligned}$$

and thus

$$|G_\omega(x, y; z)| \leqslant \frac{2c}{|u_x u_y - c^2|} = \frac{2c^{-1}}{|c^{-2} u_x u_y - 1|}$$

We note that for any $w_x, w_y \in \mathbb{R}$

$$\min(|w_x - f(w_y)|, |w_y - f(w_x)|) \leqslant |w_x w_y - 1| \tag{10}$$

where $wf(w) = \min(1, w^2)$. Indeed, if $w_x^2 \geqslant 1$, then

$$|w_y - f(w_x)| = |w_y - w_x^{-1}| \leqslant |w_x w_y - 1|$$

and the same argument applies if $w_y^2 \geqslant 1$. If, however, $w_x^2, w_y^2 < 1$, then

$$\begin{aligned}[w_x - f(w_y)]^2 = [w_y - f(w_x)]^2 &= (w_x - w_y)^2 \\ &= (w_x w_y - 1)^2 - (1 - w_x^2)(1 - w_y^2) < (w_x w_y - 1)^2\end{aligned}$$

By (10) we estimate

$$|G_\omega(x, y; z)|^s \leqslant 2^s(|u_x - cf(c^{-1}u_y)|^{-s} + |u_y - cf(c^{-1}u_x)|^{-s})$$

To estimate its expectation we shall use that

$$\begin{aligned}\int dv\, \rho(v)\, |v-\beta|^{-s} &\leqslant \lambda^{-s} \int_{|v-\beta| \geqslant \lambda} dv\, \rho(v) + \|\rho\|_\infty \int_{|v-\beta|<\lambda} dv\, |v-\beta|^{-s} \\ &\leqslant \lambda^{-s} \|\rho\|_1 + \frac{2\lambda^{1-s}}{1-s} \|\rho\|_\infty \leqslant C_s \|\rho\|_1^{1-s} \|\rho\|_\infty^s \end{aligned} \tag{11}$$

with $C_s = (2/s)^s (1-s)^{-1}$ after minimizing over $\lambda > 0$. (This estimate holds for any $\beta \in \mathbb{C}$ although we use it here for $\beta \in \mathbb{R}$). Hence

$$\int dv_x\, dv_y\, \rho(v_x)\, \rho(v_y)\, |G_\omega(x, y; z)|^s \leqslant 2 \cdot 2^s C_s \|\rho\|_\infty^s$$

(ii) In case (9) fails, we have $|\mathrm{Im}\, a_{ii}| \geqslant |a_{xy}|/2$ for $i = x$ or $i = y$. We shall consider only $i = y$, the other case being similar. Then

$$|G_\omega(x, y; z)| \leqslant \frac{2}{|u_x + [u_y \,\mathrm{Im}\, a_{xx} - \mathrm{Im}(a_{xy} a_{yx})](\mathrm{Im}\, a_{yy})^{-1}|}$$

$$\int dv_x\, dv_y\, \rho(v_x)\, \rho(v_y)\, |G_\omega(x, y; z)|^s \leqslant 2^s C_s \|\rho\|_\infty^s$$

By joining the results of the two cases, we see that the expectation with respect to v_x, v_y is bounded uniformly in $\hat{\omega}$. ∎

Similarly, the next tool is a version of the decoupling lemma of ref. 1.

Lemma 6. Let $0<s<1$. Then there is $c>0$ such that

$$\frac{\int dv\, \rho(v)(|v-\eta|^s/|v-\beta|^s)}{\int dv\, \rho(v)(1/|v-\beta|^s)} \geqslant c\, \frac{\|\rho\|_1^s}{\|\rho\|_\infty^s} \tag{12}$$

for all $\rho\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$, $0\not\equiv\rho\geqslant 0$, and all β, $\eta\in\mathbb{C}$.

Proof. We may assume $\eta\in\mathbb{R}$ since the integral in the numerator becomes smaller upon replacing η by its real part. By translation we may then assume $\eta=0$. Finally, by scaling we may assume $\|\rho\|_1=\|\rho\|_\infty=1$. We then write N (resp. D) for the numerator (resp. denominator) of the fraction in (12) and distinguish between the cases (i) $\int_{|v|\geqslant|\beta|} dv\,\rho(v)\geqslant 1/2$ and (ii) $\int_{|v|<|\beta|} dv\,\rho(v)>1/2$.

(i) In this case,

$$N\geqslant\int_{|v|\geqslant|\beta|} dv\,\rho(v)\,\frac{|v|^s}{|v-\beta|^s}\geqslant 2^{-s}\int_{|v|\geqslant|\beta|} dv\,\rho(v)\geqslant 2^{-(s+1)}$$

and $D\leqslant C_s$ by (11).

(ii) For any $\lambda>0$

$$\int_{|v|\leqslant\lambda} dv\,\rho(v)\,\frac{1}{|v-\beta|^s}\leqslant\int_{|v|\leqslant\lambda} dv\,\frac{1}{|v-\beta|^s}$$
$$\leqslant\min\left(\frac{2\lambda}{(|\beta|-\lambda)_+^s},\frac{2\lambda^{1-s}}{1-s}\right)\leqslant \text{const}\cdot\lambda\,|\beta|^{-s}$$

so that

$$N\geqslant\lambda^s\int_{|v|>\lambda} dv\,\rho(v)\,\frac{1}{|v-\beta|^s}\geqslant\lambda^s(D-\text{const}\cdot\lambda\,|\beta|^{-s})$$

Since

$$D\geqslant\int_{|v|<|\beta|} dv\,\rho(v)\,\frac{1}{|v-\beta|^s}\geqslant(2\,|\beta|)^{-s}\int_{|v|<|\beta|} dv\,\rho(v)>\frac{1}{2}(2\,|\beta|)^{-s}$$

we find $N/D\geqslant\lambda^s(1-c\lambda)\geqslant$ const for some constant c and $\lambda=(2c)^{-1}$. ∎

Proof of Lemma 2.[1] According to (7), we have

$$G_\omega(x,y;z)=\frac{\alpha}{v_y-\beta} \tag{13}$$

where α, β depend on $\{v_i\}_{i \neq y}$, but not on v_y. By taking the xy matrix element of $(h_\omega - z)^{-1}(h_\omega - z) = \mathbb{1}$, we obtain for $y \neq x$

$$\sum_{|e|=1} G_\omega(x, y+e; z) = (v_y - z)\, G_\omega(x, y; z)$$

and hence

$$\sum_{|e|=1} |G_\omega(x, y+e; z)|^s \geqslant |v_y - z|^s\, |G_\omega(x, y; z)|^s$$

We then take expectations using (13), (12),

$$\left\langle \sum_{|e|=1} |G_\omega(x, y+e; z)|^s \right\rangle \geqslant \langle |v_y - z|^s\, |G_\omega(x, y; z)|^s \rangle$$
$$\geqslant c\, \|\rho\|_\infty^{-s} \langle |G_\omega(x, y; z)|^s \rangle$$

If $y + e \neq x$ for $|e| = 1$, this can be iterated. More precisely, it can be iterated $|x - y|$ times and the terms generated can be estimated by (5):

$$\langle |G_\omega(x, y; z)|^s \rangle \leqslant (c^{-1} \|\rho\|_\infty^s)^{|x-y|} \sum_{i=1}^{(2d)^{|x-y|}} \langle |G_\omega(x, y^{(i)}; z)|^s \rangle$$
$$\leqslant (2dc^{-1} \|\rho\|_\infty^s)^{|x-y|}\, C\, \|\rho\|_\infty^s = C\, \|\rho\|_\infty^s\, e^{-m|x-y|}$$

with $e^{-m} = 2dc^{-1} \|\rho\|_\infty^s$. If $\|\rho\|_\infty$ is small enough, we have $m > 0$. ∎

Proof of Lemma 3. We consider the Hamiltonian[11] obtained from h_ω by wiggling the potential at x, namely

$$h_{\omega,\kappa} = h_\omega + \kappa P_x = h_{\omega + \kappa\delta_x}$$

The space $\Omega \times \mathbb{R} \ni (\omega, \kappa)$ is given the probability measure $d\tilde{P}(\omega, \kappa) = \rho(v_x + \kappa)\, d\kappa\, dP(\omega)$. As a result expectations related to h_ω and to $h_{\omega,\kappa}$ are the same. That is, for any P-measurable function f on Ω

$$\int dP(\omega)\, f(\omega) = \int d\tilde{P}(\omega, \kappa)\, f(\omega + \kappa\delta_x) \tag{14}$$

By the resolvent identity $(h_\omega - z)^{-1} = [1 + \kappa(h_\omega - z)^{-1} P_x](h_{\omega,\kappa} - z)^{-1}$ we have

$$G_{\omega,\kappa}(x, y; z) = \frac{G_\omega(x, y; z)}{1 + \kappa G_\omega(x, x; z)} = \frac{1}{\kappa + G_\omega(x, x; z)^{-1}} \cdot \frac{G_\omega(x, y; z)}{G_\omega(x, x; z)}$$

for all $y \in \mathbb{Z}^d$. In particular, since $|G_{\omega,\kappa}(x, x; z)| \leqslant |\text{Im } z|^{-1}$ for all $\kappa \in \mathbb{R}$ we have $|\text{Im } G_\omega(x, x; z)^{-1}| \geqslant |\text{Im } z|$. Thus

$$|\text{Im } z| \cdot |G_{\omega,\kappa}(x, y; z)|^2 \leqslant \frac{|\text{Im } G_\omega(x, x; z)^{-1}|}{|\kappa + G_\omega(x, x; z)^{-1}|^2} \cdot \frac{|G_\omega(x, y; z)|^2}{|G_\omega(x, x; z)|^2}$$

On the other hand, we also have

$$\begin{aligned} |\text{Im } z| \cdot |G_{\omega,\kappa}(x, y; z)|^2 &\leqslant |\text{Im } z| \sum_{y' \in \mathbb{Z}^d} |G_{\omega,\kappa}(x, y'; z)|^2 \\ &= |\text{Im } z| \, (\delta_x, (h_{\omega,\kappa} - z)^{-1} (h_{\omega,\kappa} - \bar{z})^{-1} \delta_x) \\ &= |\text{Im } G_{\omega,\kappa}(x, x; z)| = \frac{|\text{Im } G_\omega(x, x; z)^{-1}|}{|\kappa + G_\omega(x, x; z)^{-1}|^2} \end{aligned}$$

Let $0 < s < 1$. Using that $\min(1, t^2) \leqslant t^s$ for $t \geqslant 0$, we combine the above two estimates as

$$|\text{Im } z| \cdot |G_{\omega,\kappa}(x, y; z)|^2 \leqslant \frac{|\text{Im } G_\omega(x, x; z)^{-1}|}{|\kappa + G_\omega(x, x; z)^{-1}|^2} \cdot \frac{|G_\omega(x, y; z)|^s}{|G_\omega(x, x; z)|^s}$$

We then claim that

$$\sup_{\substack{w \in \mathbb{C} \\ v_x \in \text{supp } \rho}} |\text{Im } w| \cdot |w|^s \int d\kappa \, \rho(v_x + \kappa) \frac{1}{|\kappa + w|^2} < +\infty \tag{15}$$

so that upon using (14) and (3) we obtain

$$|\text{Im } z| \, \langle |G_\omega(x, y; z)|^2 \rangle \leqslant \text{const} \cdot \langle |G_\omega(x, y; z)|^s \rangle \leqslant \text{const} \cdot e^{-m|x-y|}$$

To prove (15) we note that by $|w|^s \leqslant |\kappa|^s + |\kappa + w|^s$ we need to estimate

$$\begin{aligned} &|\text{Im } w| \int d\kappa \, \rho(v_x + \kappa) \, |\kappa|^s \frac{1}{|\kappa + w|^2} \\ &\qquad \leqslant \pi \, \| \, |\kappa|^s \rho(v_x + \kappa) \|_\infty \leqslant \pi(|v_x|^s \, \|\rho\|_\infty + \| \, |\lambda|^s \rho(\lambda) \|_\infty) \end{aligned}$$

and

$$\begin{aligned} &|\text{Im } w| \int d\kappa \, \rho(v_x + \kappa) \frac{1}{|\kappa + w|^{2-s}} \\ &\qquad \leqslant \min(|\text{Im } w|^{-(1-s)}, \text{const} \cdot \|\rho\|_\infty \, |\text{Im } w|^s) = \text{const} \cdot \|\rho\|_\infty^{1-s} \qquad \blacksquare \end{aligned}$$

Proof of Theorem 1. We first prove (1). By Wiener's theorem (see, e.g., ref. 3) we have for any states φ, ψ

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t ds\, |(\varphi, e^{-ihs}\psi)|^2 = \sum_{\lambda\in\mathbb{R}} |(\varphi, E(\{\lambda\})\psi)|^2$$

where $E(\cdot)$ is the projection-valued measure associated with h. Using $P_{|x|<R} = \sum_{|x|<R} \delta_x(\delta_x, \cdot)$, this yields

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t}\int_0^t ds\, \|P_{|x|\geqslant R}e^{-ihs}\psi\|^2 \\
&= \|\psi\|^2 - \lim_{t\to\infty}\frac{1}{t}\int_0^t ds\, \|P_{|x|<R}e^{-ihs}\psi\|^2 \\
&= \|\psi\|^2 - \sum_{\lambda\in\mathbb{R}} \|P_{|x|<R}E(\{\lambda\})\psi\|^2 \\
&= \|E_c\psi\|^2 + \sum_{\lambda\in\mathbb{R}} [\|E(\{\lambda\})\psi\|^2 - \|P_{|x|<R}E(\{\lambda\})\psi\|^2] \\
&= \|E_c\psi\|^2 + \sum_{\lambda\in\mathbb{R}} \|P_{|x|\geqslant R}E(\{\lambda\})\psi\|^2
\end{aligned}$$

from which (1) follows. This in turn implies (2) by means of an Abelian limit and of Parseval's identity. If $I\subset\mathbb{R}$ is a compact set containing the spectrum $\sigma(h)$ in its interior, we have

$$\begin{aligned}
&\varepsilon\int_{\mathbb{R}\setminus I} dE\, \|P_{|x|\geqslant R}(h-E-i\varepsilon)^{-1}\psi\|^2 \\
&\leqslant \varepsilon\int_{\mathbb{R}\setminus I} dE\, \|(h-E-i\varepsilon)^{-1}\psi\|^2 \\
&\leqslant \varepsilon\,\|\psi\|^2 \sup_{\lambda\in\sigma(h)}\int_{\mathbb{R}\setminus I} dE\, |\lambda-E-i\varepsilon|^{-2} \xrightarrow[\varepsilon\downarrow 0]{} 0
\end{aligned}$$

Since $\|\Delta\|\leqslant 2d$ we have $\sigma(h_\omega)\subset[-2d, 2d]+\operatorname{supp}\rho\subset I$ for some fixed compact I, with probability 1. Hence

$$\begin{aligned}
\|E_{\omega,c}\delta_0\|^2 &= \lim_{R\to\infty}\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\int_I dE\, \|P_{|x|\geqslant R}(h_\omega - E - i\varepsilon)^{-1}\delta_0\|^2 \\
&= \lim_{R\to\infty}\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\int_I dE \sum_{|x|\geqslant R} |G_\omega(x, 0; E+i\varepsilon)|^2
\end{aligned}$$

almost surely. By Fatou's lemma and (4) we conclude

$$\langle \|E_{\omega,c}\delta_0\|^2\rangle \leqslant \varliminf_{R\to\infty} \varliminf_{\varepsilon\downarrow 0} \frac{\varepsilon}{\pi}\int_I dE \sum_{|x|\geqslant R} \langle |G_\omega(x,0;E+i\varepsilon)|^2\rangle$$

$$\leqslant \frac{C\,|I|}{\pi} \varliminf_{R\to\infty} \sum_{|x|\geqslant R} e^{-m|x|}=0$$

Similarly, $E_{\omega,c}\delta_x=0$ almost surely for any $x\in\mathbb{Z}^d$, i.e., $E_{\omega,c}=0$. ∎

ACKNOWLEDGMENTS

I thank M. Aizenman, P. Hislop, and I. M. Sigal for interesting discussions and A. Jensen and D. Laksov for their hospitality at the Mittag–Leffler Institute where this work was begun. After completion of this work I received a preprint by M. Aizenman, "Localization at weak disorder: Some elementary bounds," which contains results related to ours. I thank the author for informing me. I gratefully acknowledge an Alfred P. Sloan Fellowship.

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Communicated by M. Aizenman